MATH 1920, Spring 2017
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Sections: 205, 206

Section 15.4–15.5 Main Points

Section 15.4: Differentiability and Tangent Planes

Definition (Differentiability): Assume that $f(x,y)$ is defined in a disk D containing (a,b) and that $f_x(a,b)$ and $f_y(a,b)$ exist. Then, $f(x,y)$ is *differentiable* at (a,b) if it is *locally linear*– that is if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

In this case, the *tangent plane* to the graph at $(a,b,f(a,b))$ is $z = L(x,y)$ given by

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).}$$

Note that from the equation above we can see that a **normal vector** to the tangent plane at (a,b) is given by

$$\boxed{n = \langle f_x(a,b), f_y(a,b), -1\rangle.}$$

Since the local linearity condition can be hard to check, we have a theorem to help us determine differentiability.

Theorem 2. (p.785): If $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous on an open disk D, then $f(x,y)$ is differentiable on D.

By definition, if $f(x,y)$ is differentiable at (a,b) then it is locally linear and we can approximate $f(x,y)$ near a point $P = (a,b)$ by the equation of the plane tangent to f at P:

$$\boxed{f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \text{ for } (x,y) \text{ near } (a,b).}$$

Moreover, the linear approximation can be written in terms of change in f :

$$\boxed{\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y.}$$

Section 15.5: The Gradient and Directional Derivatives

Definition: The *gradient* of a function f is the vector of partial derivatives:

$$\boxed{\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.}$$

Theorem 2. Chain rule for paths (p.792): If f and $r(t)$ are differentiable, then

$$\boxed{\frac{d}{dt} f(r(t)) = \nabla f_{r(t)} \cdot r'(t).}$$

The idea of composite functions of this type can be seen as follows: If $f(x,y,z)$ gives the temperature as a function of position, and $r(t)$ gives the position of say a mountain climber as a function of time, then $\frac{d}{dt} f(r(t))$ gives the rate of change of temperature with respect to the mountain climber.

Recall that the directional derivative $D_u f(a,b)$ is the rate of change of f along the linear path through $P = (a,b)$ in the direction of unit vector u.

Theorem 3. (p.795) We can compute the directional derivative as follows: if u is a unit vector, then

$$\boxed{D_u f(P) = \nabla f_P \cdot u}$$

The following theorem is **very important for you to learn and understand** because it describes the properties of the gradient.

Theorem 4 (p.797): Assume that $\nabla f_P \neq 0$. Let u be a unit vector making an angle of θ with ∇f_P. Then,

$$\boxed{D_u f(P) = ||\nabla f_P|| \cos\theta}$$

- ∇f_P points in the direction of the maximum rate of increase of f at P.
- $-\nabla f_P$ points in the direction of the maximum rate of decrease of f at P.
- ∇f_P is normal to the level curve of f at P.

Another use of the gradient is in finding the normal vectors to the tangent planes to a surface with equation $F(x, y, z) = k$, where k is constant.

Theorem 5 (p.799) Let $P = (a, b, c)$ be a point on the surface given by $F(x, y, z) = k$ and assume that $\nabla F_p \neq 0$. Then, ∇F_P is a normal vector to the tangent plane to the surface at P. The tangent plane has equation:

$$\boxed{F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.}$$